Approximating chaotic billiards by surfaces whose geodesic flow is Anosov

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Configuration spaces

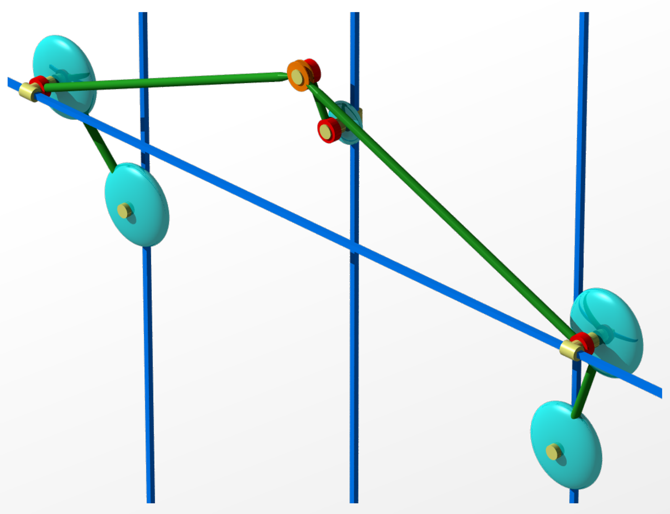

Fig. 1

The linkage on Figure 1 is one of the only physical systems whose behavior is known to be **Anosov**. All the bars are rigid, the blue discs slide along the fixed blue bars, while the green bars are mobile. Figure 2 displays the set of all the possible physical states of the system: it is a Riemannian surface X of genus 7, called the configuration space of the linkage, immersed in the product of the 2-dimensional torus with the line.

If no external force applies, when the linkage is given an initial speed, its physical evolution follows the **geodesic flow** on X. If one reduces the mass of the small central blue disk, the surface X is flattened along the vertical axis (Figure 3). It converges to a **uniformly hyperbolic billiard** (Figure 4), with very chaotic behavior. Thus, it may be shown that the geodesic flow on the surface of Figure 3 is an Anosov flow, using the fact that the projection of a geodesic onto the billiard (visible on Figure 4) is close to a billiard trajectory.

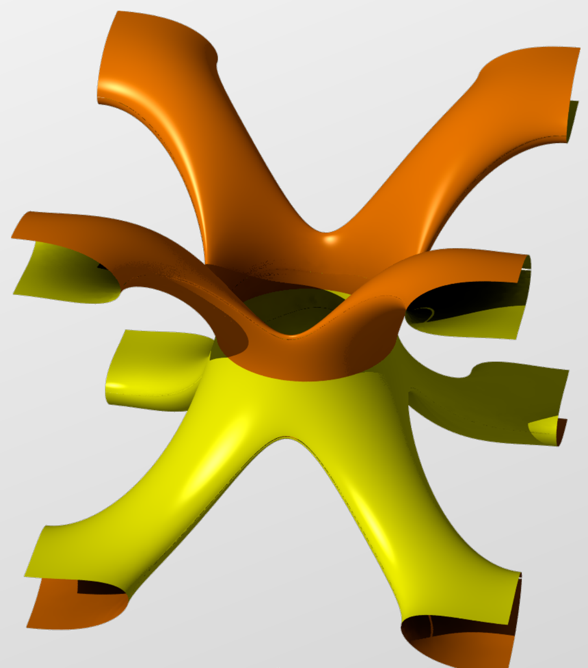

Fig. 2

Fig. 4

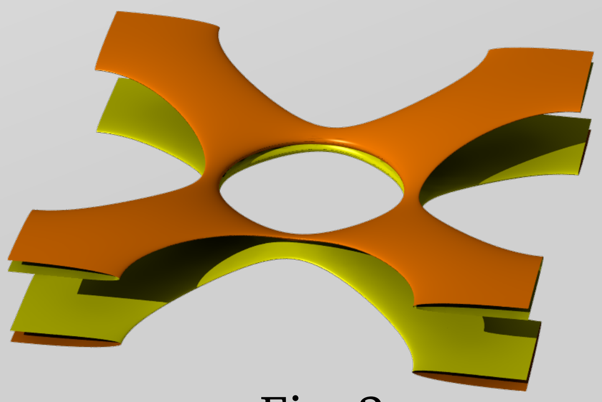

Fig. 3

Birkhoff's idea

Consider an ellipsoid and make one of its axes tend to zero: the ellipsoid flattens and tends to an ellipse in the plane formed by the two other axes. As Birkhoff had noticed, the geodesic flow on the ellipsoid converges to the billiard flow on the ellipse. In fact, this phenomenon is far more general: it applies to almost any surface in the Euclidean 3-space which is flattened along an axis.

Here, our aim is to explore the following idea: if the billiard obtained at the limit is chaotic, then the geodesic flow of the surface is also chaotic when the surface is sufficiently flattened.

Anosov geodesic flows

A geodesic flow $\phi : \mathbb{R} \times T^1M \to T^1M$ on the unit tangent bundle of a closed Riemannian manifold M is **Anosov** if there exists a decomposition of $T(T^1M)$, stable under the flow,

$$T_x(T^1M) = E^0_x \oplus E^u_x \oplus E^s_x$$

where $E^0_x = \mathbb{R}\left.\frac{d}{dt}\right|_{t=0}\phi_t(x)$, such that

$$\|D\phi^t|_{E^s_x}\| \leq a\lambda^t, \ \|D\phi^{-t}_x|_{E^u_x}\| \leq a\lambda^t$$

(for some $a > 0$ and $\lambda \in (0,1)$, which do not depend on x).

Example. Closed manifolds with negative curvature have Anosov geodesic flows.

*The approximation of uniformly hyperbolic billiards allows us to construct **new surfaces** whose geodesic flow is Anosov.*

Chaotic properties

Anosov geodesic flows and uniformly hyperbolic billiards share the following dynamical properties:

- Sensitivity to initial conditions;
- Exponential mixing (which implies, in particular, ergodicity);
- Density of periodic orbits.

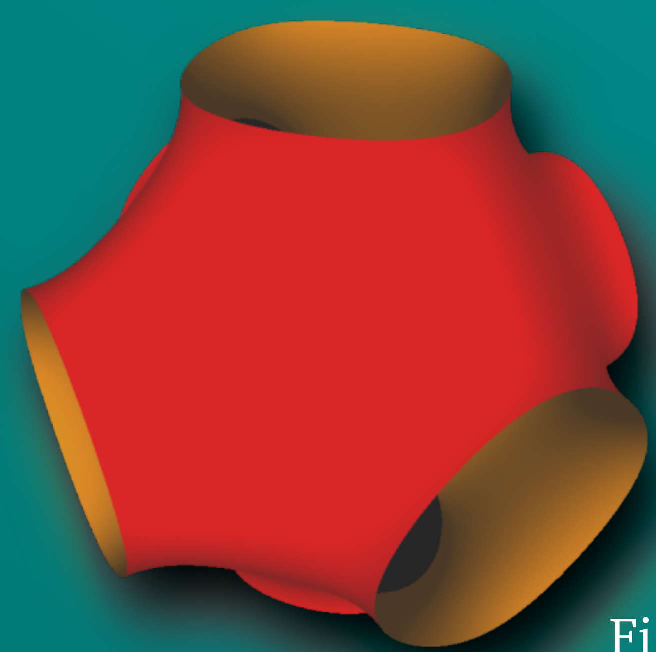

Fig. 5

Schwarz's P surface (Figure 5), introduced by Schwarz in the 1880's, has an Anosov geodesic flow (because its curvature is negative everywhere except at 8 points). It is isometrically embedded in the flat 3-torus.

*Do there exist Anosov surfaces which are **isometrically embedded** in...*
... the Euclidean 3-space? The standard 3-sphere?

Figure 6 shows the first known example of a surface of genus 11 with Anosov geodesic flow which is isometrically embedded in the **standard 3-sphere** (it is seen in stereographic projection). This surface is obtained by approximating a spherical billiard (Figure 8) which is known to be **uniformly hyperbolic**, and whose obstacles are centered at the vertices of a regular icosahedron. We do not know whether it is possible to obtain a genus smaller than 11 for such a surface.

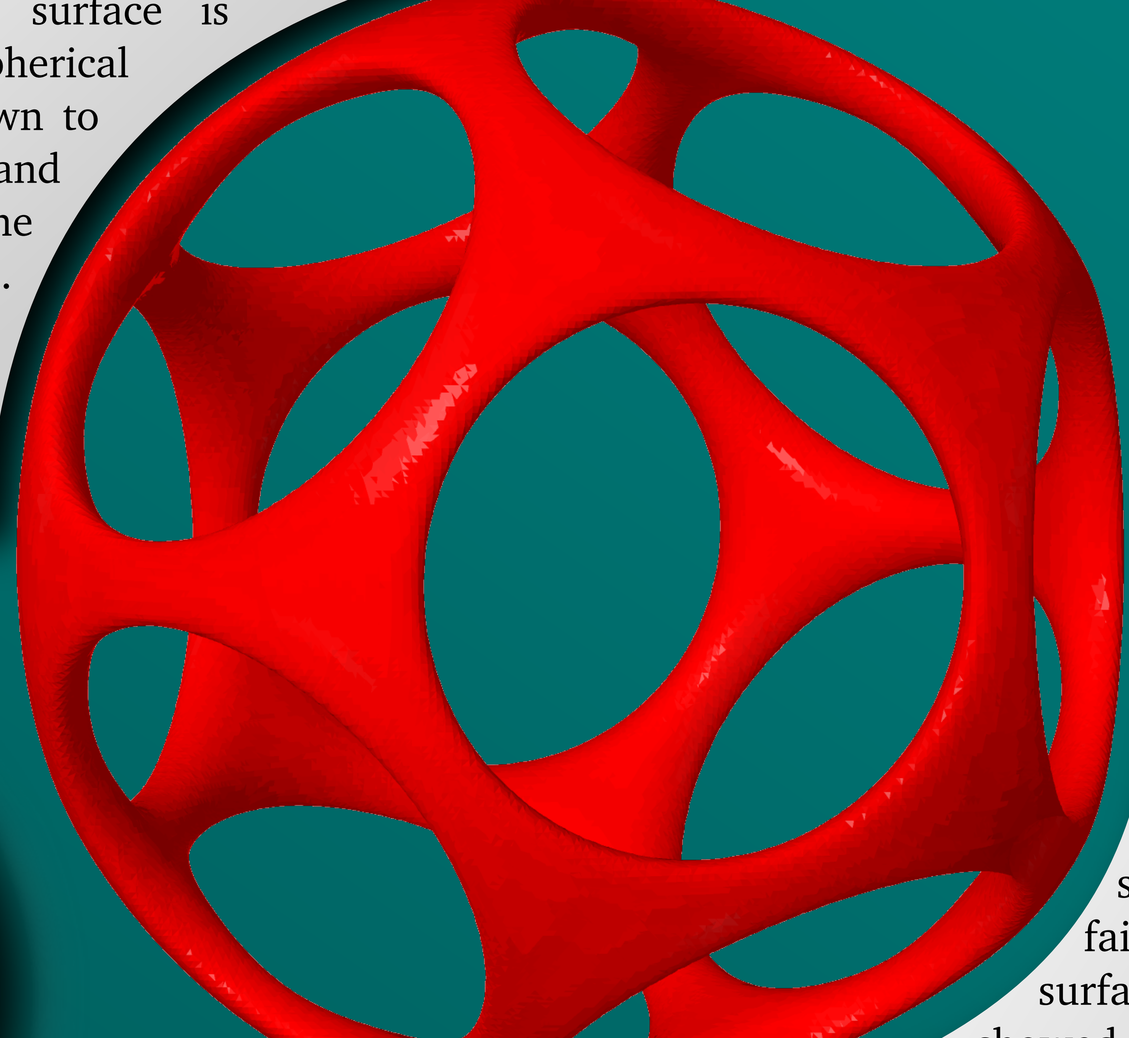

Fig. 6

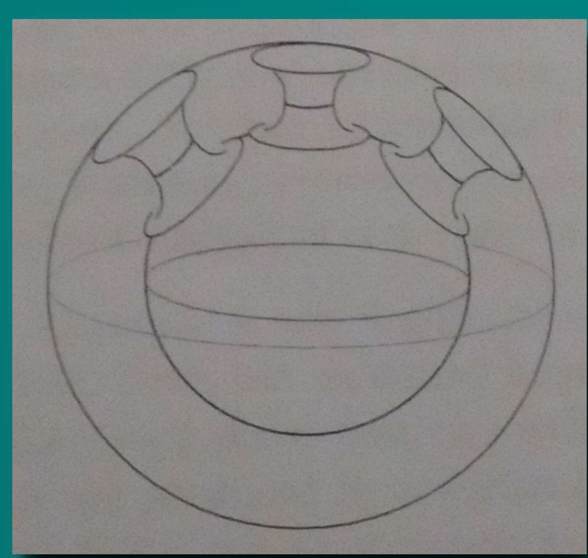

Fig. 7

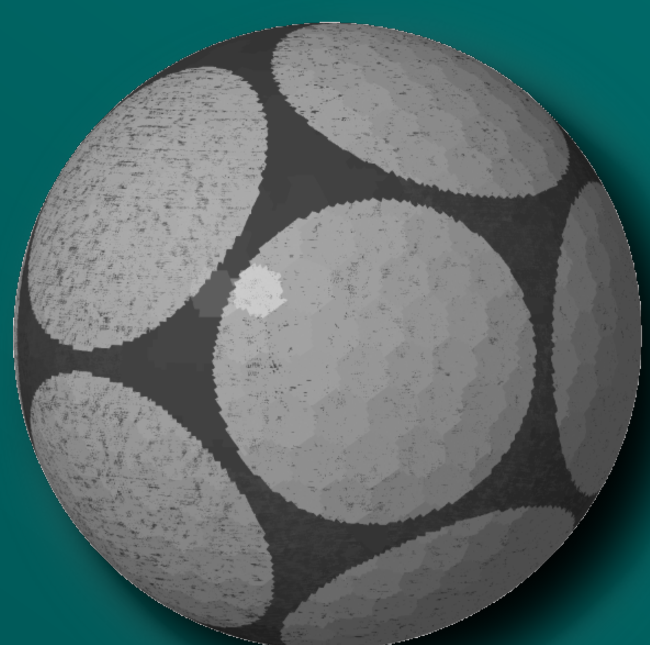

Fig. 8

Unfortunately, regarding embeddings in the **Euclidean 3-space**, it may be shown that any approximation of the spherical billiard of Figure 8 fails to provide an Anosov surface. In 2003, Donnay and Pugh showed that it is possible to embed an Anosov surface in the Euclidean 3-space (Figure 7), but their proof does not give an explicit genus for which this embedding is possible.

Uniformly hyperbolic billiards

A smooth billiard D is a closed subset of a closed Riemannian manifold, whose boundary is smooth. Its phase space is $\Omega = T^1(\mathrm{Int}(D))$. We define $\tilde{\Omega}$ as the set of $(x,v) \in \Omega$ such that the trajectory starting from (x,v) does not hit the boundary of D tangentially.
The billiard flow ϕ is **uniformly hyperbolic** if at each point $x \in \tilde{\Omega}$, there exists a decomposition of $T_x\Omega$, stable under the flow,

$$T_x\Omega = E^0_x \oplus E^u_x \oplus E^s_x$$

where $E^0_x = \mathbb{R}\left.\frac{d}{dt}\right|_{t=0}\phi^t(x)$, such that

$$\|D\phi^t_x|_{E^s_x}\| \leq a\lambda^t, \ \|D\phi^{-t}_x|_{E^u_x}\| \leq a\lambda^t$$

(for some $a > 0$ and $\lambda \in (0,1)$, which do not depend on x).

Example. A billiard in the flat torus $\mathbb{T}^2$ is uniformly hyperbolic if its boundary is curved negatively, and if every trajectory meets the boundary (like on Figure 4).

References

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I would like to thank Jos Leys for realizing Figures 1, 2, 3 and 4.